Basic Mathematics

Quadratics

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The aim of this document is to provide a short, self assessment programme for students who wish to acquire a basic competence in the theory of quadratic expressions.

Last Revision Date: February 4, 2002 Version 1.0

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1. Quadratic Expressions (Introduction)

In the package on Factorising Expressions we looked at how to factorise quadratic expressions which have the number 1 in front of the highest order term, x^2, y^2, z^2, etc.. If the highest order term has a number *other* than this then more work must be done to factorise the expression. As in the earlier case, some insight is gained by looking at a general expression with factors $(ax + c)$ and $(bx + d)$. Then

$$\begin{aligned}(ax+c)(bx+d) &= \overset{\mathbf{F}}{a}xbx + \overset{\mathbf{O}}{a}xd + \overset{\mathbf{I}}{c}bx + \overset{\mathbf{L}}{cd}\,, \\ &= abx^2 + (ad+bc)x + cd\,.\end{aligned}$$

showing that the coefficient of the square term, x^2, is ab, the product of the coefficients of the x-terms in each factor. The coefficient of the x-term is made up from the coefficients as follows:

$$\begin{array}{l}\text{(Outside of left bracket)}\times\text{(Outside of right bracket)}\\ \qquad\qquad\qquad\qquad\qquad + \\ \text{(Inside of left bracket)}\quad\times\text{(Inside of right bracket)}\end{array}$$

This is the information needed to find the factors of quadratic expressions.

Example 1 Factorise the following expressions.

$$\text{(a) } 2x^2 + 7x + 3\,, \quad \text{(b) } 10x^2 + 9x + 2\,.$$

Solution

(a) The factors of 2 are 2 and 1, and the factors of 3 are 3 and 1. If the quadratic expression factorises then it is likely to be of the form $(2x + c)(1x + d)$ and the choice for c, d is $3, 1$ or $1, 3$. Trying the first combination,

$$\begin{aligned}(2x+3)(x+1) &= 2x^2 + 2x + 3x + 3\,,\\ &= 2x^2 + 5x + 3 \text{ (which is incorrect)}\,.\end{aligned}$$

The second choice is

$$\begin{aligned}(2x+1)(x+3) &= 2x^2 + 6x + x + 3\,,\\ &= 2x^2 + 7x + 3\,,\end{aligned}$$

which is therefore the correct factorisation.

(b) There is more than one choice for the first term since 10 is 1×10 as well as 2×5. The final term will factor as 2×1. Which combination of pairs, either $(1, 10)$ with $(2, 1)$, or $(2, 5)$ with $(2, 1)$, will give the correct coefficient of x, i.e., 9? The latter two pairs seem the more likely since $2 \times 2 + 5 \times 1 = 9$. Checking

$$\begin{aligned}(2x + 1)(5x + 2) &= 10x^2 + 4x + 5x + 2\,, \\ &= 10x^2 + 9x + 2\,.\end{aligned}$$

EXERCISE 1. Factorise each of the following expressions. (Click on green letters for solutions.)

(a) $2x^2 + 5x + 3$ (b) $3x^2 + 7x + 2$
(c) $3y^2 - 5y - 2$ (d) $4z^2 - 23z + 15$
(e) $64z^2 + 4z - 3$ (f) $4w^2 - 25$

Quiz To which of the following does $12x^2 + 17x - 14$ factorise?

(a) $(12x + 7)(x - 2)$ (b) $(x + 2)(12x - 7)$
(c) $(4x + 7)(x - 3)$ (d) $(x - 7)(4x + 3)$

2. Solving Quadratic Equations

Factorising a quadratic expression and finding the roots of a quadratic equation are closely related.

Example 2 Find the solution to the equations

$$\text{(a) } x^2 + 6x + 8 = 0, \quad \text{(b) } x^2 - 4x + 4 = 0.$$

Solution

(a) The quadratic expression will factorise as follows.

$$x^2 + 6x + 8 = (x + 2)(x + 4).$$

The solution to the equation may now be obtained;

$$\begin{aligned} \text{If } \; x^2 + 6x + 8 &= 0 \\ \text{then } \; (x + 2)(x + 4) &= 0\,. \end{aligned}$$

Thus either $(x + 2) = 0$, or $(x + 4) = 0$. The solution to the equation is thus $x = -2$ or $x = -4$.

(b) In this example the expression is

$$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2\,.$$

The solution to the equation $x^2 - 4x + 4 = 0$ is thus $x = 2$.

In this case, the equation is said to have *equal roots*.

EXERCISE 2. Find the solution to each of the following equations. (Click on green letters for solutions.)

(a) $2x^2 + 5x + 3 = 0$	(b) $3x^2 + 7x + 2 = 0$
(c) $3y^2 - 5y - 2 = 0$	(d) $4z^2 - 23z + 15 = 0$
(e) $64z^2 + 4z - 3 = 0$	(f) $4w^2 - 25 = 0$

Quiz Which of the following is the solution to the quadratic equation

$$12x^2 + 17x - 14 = 0\,?$$

(a) $2, 7/12$	(b) $-2, -7/12$
(c) $-2, 7/12$	(d) $2, -7/12$

3. Complete Squares

In Example 2 (b) we encountered the quadratic expression

$$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2 \,.$$

For the obvious reason, this expression is called a *complete square*. Quadratic expressions which can be factored into a complete square are useful in many situations. They have a particularly simple structure and it is important to be able to recognise such factorisations.

Example 3 Show that the following quadratic expressions are complete squares.

(*a*) $x^2 + 6x + 9\,,$ (*b*) $x^2 + 4x + 4,$
(*c*) $x^2 - 2x + 1,$ (*d*) $x^2 - 2ax + a^2.$

Solution

In each of these cases it is easy to check the following.

(*a*) $x^2 + 6x + 9 = (x+3)^2\,,$ (*b*) $x^2 + 4x + 4 = (x+2)^2\,,$
(*c*) $x^2 - 2x + 1 = (x-1)^2\,,$ (*d*) $x^2 - 2ax + a^2 = (x-a)^2.$

The last example, $x^2 - 2ax + a^2 = (x - a)^2$, is a general case and it may be used to find perfect squares for any given example. It may be usefully employed in finding solutions to the following exercises.

EXERCISE 3. Write each of the following as a complete square. (Click on green letters for solutions.)

(a) $x^2 - 10x + 25$ (b) $z^2 + 8z + 16$
(c) $w^2 - w + 1/4$ (d) $y^2 + 5y + 25/4$

Quiz Which of the following quadratic expressions is a complete square?

(a) $z^2 + 3z - 9/4$ (b) $z^2 + 3z + 9/2$
(c) $z^2 - 3z + 9/4$ (d) $z^2 - 3z + 9/2$

4. Completing the Square

Not every quadratic is a complete square but it is possible to write **ALL** quadratics as a complete square plus a number. This is the process known as *COMPLETING THE SQUARE*. This is an extremely useful algebraic procedure with many applications.

As an example, let us take the quadratic expression $x^2 - 4x + 5$. From the beginning of Section 3 we note that

$$x^2 - 4x + 4 = (x - 2)^2 \,.$$

Since

$$x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 \,,$$

the expression may be written as

$$x^2 - 4x + 5 = (x - 2)^2 + 1 \,,$$

and we have *completed the square on the quadratic*.

On the next page we shall look at some more examples of completing the square.

Example 4 Use the results of Example 3 to complete the square on each of the following quadratic expressions.

(*a*) $x^2 + 6x + 11$, (*b*) $x^2 + 4x + 3$,
(*c*) $2x^2 + 8x + 4$, (*d*) $x^2 - 2ax + a^2 + b^2$.

Solution

(a) Since $x^2 + 6x + 11 = (x^2 + 6x + 9) + 2$, the expression may be written $x^2 + 6x + 11 = (x + 3)^2 + 2$.

(b) Here we note that $x^2 + 4x + 3 = (x^2 + 4x + 4) - 1$. Now we may use the fact that $(x + 2)^2 = (x^2 + 4x + 4)$, to obtain $x^2 + 4x + 3 = (x^2 + 4x + 4) - 1 = (x + 2)^2 - 1$.

(c) Since $2x^2 + 8x + 4 = 2(x^2 + 4x + 2) = 2(\{x^2 + 4x + 4\} - 2)$, we have $2x^2 + 8x + 4 = 2(\{x + 2\}^2 - 2) = 2(x + 2)^2 - 4$.

(d) Here $x^2 - 2ax + a^2 + b^2 = (x - a)^2 + b^2$.

The starting position for each of the above cases was the *correct choice* of the complete square. For example, part (a) began with the expansion of $(x+3)^2$ as x^2+6x+9 and this information was used to complete the square on $x^2+6x+10$. In practice the starting point is usually the quadratic expression, i.e., in (a) it would be $x^2+6x+10$. The problem is to work directly from this. The general procedure uses the following observation:

$$\left(x+\frac{p}{2}\right)^2 = x^2+px+\left(\frac{p}{2}\right)^2 ,$$

so the expression x^2+px can be made into a complete square by adding $(p/2)^2$, i.e., *by adding the square of half the coefficient of x*. So as not to alter the expression *the same amount must be subtracted*. In other words we use the equality

$$x^2+px = \left(x+\frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 ,$$

and this is the essence of completing the square.

Example 5 Complete the square on the following expressions.

(a) $x^2 + 8x + 15\,,$ (b) $x^2 - 5x + 6\,.$

(a) Completing the square means *adding the square of half the coefficient of x* and then *subtracting the same amount*. Thus

$$\begin{aligned}
x^2 + 8x + 15 &= [x^2 + 8x] + 15\,, \\
&= \left[x^2 + 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2\right] + 15\,, \\
&= \left[(x+4)^2 - 4^2\right] + 15\,, \\
&= (x+4)^2 - 16 + 15\,, \\
&= (x+4)^2 - 1\,.
\end{aligned}$$

(b) For $x^2 - 5x + 6$ the procedure is much the same.

$$\begin{aligned}
x^2 - 5x + 6 &= [x^2 - 5x] + 6\,, \\
&= \left[\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2\right] + 6\,, \\
&= \left[\left(x - \frac{5}{2}\right)^2 - \frac{25}{4}\right] + 6\,, \\
&= \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{24}{4}\,, \\
&= \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}\,.
\end{aligned}$$

Some exercises for practice may be found on the next page.

Exercise 4. Complete the square on each of the following. (Click on green letters for solutions.)

(a) $x^2 - 6x + 5$ (b) $2z^2 + 8z + 9$
(c) $2w^2 - 5w + 7$ (d) $3y^2 + 2y + 2$

Quiz Which of the expressions below is obtained after completing the square on $2x^2 - 3x + 5$?

(a) $2\left(x - \frac{3}{2}\right)^2 + \frac{21}{4}$ (b) $2\left(x - \frac{3}{4}\right)^2 + \frac{21}{8}$
(c) $2\left(x - \frac{3}{2}\right)^2 + \frac{31}{8}$ (d) $2\left(x - \frac{3}{4}\right)^2 + \frac{31}{8}$

5. Quiz on Quadratics

Begin Quiz In each of the following, choose:

1. the factorisation of $6y^2 + 7y + 2$

(a) $(2y+1)(3y+2)$ (b) $(2y+2)(y+3)$
(c) $(6y+1)(y+2)$ (d) $(6y+2)(y+1)$

2. the perfect square

(a) $x^2 + 9x - 81$ (b) $x^2 - 9x + 81$
(c) $x^2 + 18x - 81$ (d) $x^2 - 18x + 81$

3. the roots of $2w^2 + 3w - 20 = 0$

(a) 5/2, -4 (b) -4/2, 5 (c) -5/2, 4 (d) 4/2, -5

4. the result of completing the square on $2x^2 + 16x + 49$

(a) $2(x+2)^2 + 19$ (b) $2(x+4)^2 + 17$
(c) $2(x+8)^2 + 17$ (d) $2(x+8)^2 + 19$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) In this case we have

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

Click on green square to return

Exercise 1(b) In this case we have

$$3x^2 + 7x + 2 = (3x + 1)(x + 2)$$

Click on green square to return

Exercise 1(c) In this case we have

$$3y^2 - 5y - 2 = (3y + 1)(y - 2)$$

Click on green square to return □

Exercise 1(d) In this case we have

$$4z^2 - 23z + 15 = (4z - 3)(z - 5)$$

Click on green square to return $\square$

Exercise 1(e) In this case we have

$$64z^2 + 4z - 3 = (16z - 3)(4z + 1)$$

Click on green square to return □

Exercise 1(f) This is a case of the difference of two squares which was seen in the package on Brackets.

$$4w^2 - 25 = (2w - 5)(2w + 5)$$

Click on green square to return $\square$

Exercise 2(a) In this case we have, from Exercise 1 (a), that

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)\,.$$

Thus if $2x^2 + 5x + 3 = 0$ then we have

$$\text{either } (2x + 3) = 0\,, \text{ or } (x + 1) = 0.$$

For the first of these

$$\begin{aligned} 2x + 3 &= 0 \\ 2x &= -3 \text{ (adding } -3 \text{ to both sides)} \\ x &= -\frac{3}{2} \text{ (dividing both sides by } 2)\,. \end{aligned}$$

The solution to the second is obviously $x = -1$.

Click on green square to return

Exercise 2(b) In this case we have, from Exercise 1 (b), that

$$\begin{array}{rrcl} & 3x^2+7x+2 & = & (3x+1)(x+2)\,, \\ \text{so that if} & 3x^2+7x+2 & = & 0\,, \\ \text{then} & (3x+1)(x+2) & = & 0\,. \end{array}$$

$$\text{Thus either}\quad 3x+1=0 \quad \text{or} \quad x+2=0\,.$$

For the first of these

$$\begin{array}{rcl} 3x+1 & = & 0 \\ 3x & = & -1 \text{ (adding } -1 \text{ to both sides)} \\ x & = & -\dfrac{1}{3} \text{ (dividing both sides by } 3)\,. \end{array}$$

The solution to the second part is obviously $x=-2$. The solution to the original equation is thus $x=-2$ or $x=-1/3$.

Click on green square to return

Exercise 2(c) In this case we have, from Exercise 1 (c), that

$$3y^2 - 5y - 2 = (3y + 1)(y - 2) = 0\,.$$

Thus either $3y + 1 = 0$, or $y - 2 = 0$. For the first part,

$$\begin{aligned} 3y + 1 &= 0\,, \\ 3y &= -1 \text{ (adding } -1 \text{ to both sides)}\,, \\ y &= -\frac{1}{3} \text{ (dividing both sides by } 3)\,. \end{aligned}$$

The solution to the second part is obviously $y = 2$.
The quadratic equation $3y^2 - 5y - 2 = 0$ thus has the solution
$y = -1/3$ or $y = 2$.

Click on green square to return □

Exercise 2(d) In this case we have, from Exercise 1 (d), that

$$4z^2 - 23z + 15 = (4z - 3)(z - 5) = 0\,.$$

Thus either $4z - 3 = 0$, or $z - 5 = 0$.

Proceeding as in the previous examples, the solution to the first part is $z = 3/4$ and to the second part is $z = 3$.

The solution to $4z^2 - 23z + 15 = 0$ is therefore $z = 3/4$ or $z = 3$.

Click on green square to return □

Exercise 2(e) In this case we have, from Exercise 1 (e), that

$$64z^2 + 4z - 3 = (16z - 3)(4z + 1) = 0\,.$$

Thus either $16z - 3 = 0$ or $4z + 1 = 0$. For the first part

$$\begin{aligned} 16z - 3 &= 0\,, \\ 16z &= 3 \text{ (adding } 3 \text{ to both sides)}\,, \\ z &= \frac{3}{16} \text{ (dividing both sides by } 16)\,. \end{aligned}$$

For the second part

$$\begin{aligned} 4z + 1 &= 0 \\ 4z &= -1 \text{ (adding } -1 \text{ to both sides)}\,, \\ z &= -\frac{1}{4} \text{ (dividing both sides by } 4)\,. \end{aligned}$$

The solution to the equation $64z^2 + 4z - 3 = 0$ is thus
$z = 3/16$ or $z = -1/4$.

Click on green square to return □

Exercise 2(f) In this case we have, from Exercise 1 (f), that

$$4w^2 - 25 = (2w - 5)(2w + 5) = 0\,.$$

The solution to this is

$$w = 5/2 \text{ or } w = -5/2\,,$$

$$\text{i.e.} \quad w = \pm 5/2\,.$$

Click on green square to return □

Exercise 3(a) Comparing $x^2 - 2ax + a^2$ with $x^2 - 10x + 25$ we see that $2a = 10$, and $a^2 = 25$. Thus $a = 5$ is the solution. It may easily be checked that

$$(x - 5)^2 = x^2 - 10x + 25\,.$$

Click on green square to return

□

Exercise 3(b) Comparing $z^2 - 2az + a^2$ with $z^2 + 8z + 16$, it follows that $-2a = 8$, and $a^2 = 16$. In this case $a = -4$. Checking

$$(z - (-4))^2 = (z + 4)^2 = z^2 + 8z + 16\,.$$

Click on green square to return

Exercise 3(c) Comparing $w^2 - 2aw + a^2$ with $w^2 - w + 1/4$, we must have $-2a = -1$, and $a^2 = 4$. In this case $a = 1/2$, and

$$a^2 = \left(\frac{1}{2}\right)^2 = \frac{(1)^2}{(2)^2} = \frac{1}{4}\,.$$

It is now easy to check that

$$\left(w - \frac{1}{2}\right)^2 = w^2 - w + \frac{1}{4}\,.$$

Click on green square to return □

Exercise 3(d) Comparing $y^2 - 2ay + a^2$ with $y^2 + 5y + 25/4$ we must have $-2a = 5$, and $a^2 = 25/4$. From this it must follow that $a = -5/2$. Note that

$$a^2 = \left(-\frac{5}{2}\right)^2 = \frac{(-5)^2}{(2)^2} = \frac{25}{4}\,.$$

It is now easy to check that

$$\left(y - \left(-\frac{5}{2}\right)\right)^2 = \left(y + \frac{5}{2}\right)^2 = y^2 + 5y + \frac{25}{4}\,.$$

Click on green square to return □

Exercise 4(a) Here

$$\begin{aligned} x^2 - 6x + 5 &= [x^2 - 6x] + 5 \\ &= [x^2 - 6x + (3)^2 - (3)^2] + 5 \\ &= [(x-3)^2 - 9] + 5 \\ &= (x-3)^2 - 4 \end{aligned}$$

Click on green square to return

Exercise 4(b)

$$\begin{aligned}
2z^2 + 8z + 9 &= 2[z^2 + 4z] + 9 \\
&= 2[z^2 + 4z + (2)^2 - (2)^2] + 9 \\
&= 2[(z+2)^2 - 4] + 9 \\
&= 2(z+2)^2 + 1
\end{aligned}$$

Click on green square to return

$\square$

Exercise 4(c)

$$\begin{aligned}
2w^2 - 5w + 7 &= 2\left[w^2 - \frac{5}{2}w\right] + 7 \\
&= 2\left[w^2 - \frac{5}{2} + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right] + 7 \\
&= 2\left[\left(w - \frac{5}{4}\right)^2 - \frac{25}{16}\right] + 7 \\
&= \left[2\left(w - \frac{5}{4}\right)^2 - \frac{25}{8}\right] + \frac{56}{8} \\
&= 2\left(w - \frac{5}{4}\right)^2 + \frac{31}{8}
\end{aligned}$$

Click on green square to return □

Exercise 4(d)

$$
\begin{aligned}
3y^2+2y+2 &= 3\left[y^2+\frac{2}{3}y\right]+2 \\
&= 3\left[y^2+\frac{2}{3}y+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2\right]+2 \\
&= 3\left[\left(y+\frac{1}{3}\right)^2-\frac{1}{9}\right]+2 \\
&= \left[3\left(y+\frac{1}{3}\right)^2-\frac{1}{3}\right]+\frac{6}{3} \\
&= 3\left(y+\frac{1}{3}\right)^2+\frac{5}{3}
\end{aligned}
$$

Click on green square to return □

Solutions to Quizzes

Solution to Quiz: There are several possibilities since the final term is -14 and the two quantities corresponding to c and d must therefore have opposite signs. The possible factors of 12 are $(1, 12)$, $(2, 6)$, $(3, 4)$. For -14, the possible factors are $(\pm 1, \mp 14)$, $(\pm 2, \mp 7)$. It is now a matter of trial and error. The possible combinations are

$$\begin{array}{ccccccc} (1,12) & \text{and} & (\pm 1, \mp 14)\,, & & (1,12) & \text{and} & (\pm 2, \mp 7)\,, \\ (2,6) & \text{and} & (\pm 1, \mp 14)\,, & & (2,6) & \text{and} & (\pm 2, \mp 7)\,, \\ (3,4) & \text{and} & (\pm 1, \mp 14)\,, & & (3,4) & \text{and} & (\pm 2, \mp 7)\,. \end{array}$$

By inspection $(2 \times 12) + (1 \times \{-7\}) = 24 - 7 = 17$, so the factors appear to be $(x + 2)$ and $(12x - 7)$. This can easily be checked.

$$\begin{aligned} (x+2)(12x-7) &= 12x^2 - 7x + 24x - 14\,, \\ &= 12x^2 + 17x - 14\,, \end{aligned}$$

and the required factorisation has been achieved. End Quiz

Solution to Quiz: This quadratic is the one that occurs in the first quiz. There it was seen that $12x^2 + 17x - 14 = (x + 2)(12x - 7)$, so

$$\text{either} \quad x + 2 = 0 \text{ or } 12x - 7 = 0\,.$$

The solution to the first is $x = -2$ and to the second is $x = 7/12$.

End Quiz

Solution to Quiz: Comparing $z^2 - 3z + 9/4$ with the general form of a complete square $(z-a)^2 = z^2 - 2az + a^2$

$$\text{if} \quad -2a = -3 \quad \text{then} \quad a = \frac{(-3)}{(-2)} = \frac{3}{2}\,,$$

and

$$a^2 = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4}\,.$$

Now it is easily checked that

$$\left(z - \frac{3}{2}\right)^2 = z^2 - 3z + \frac{9}{4}\,.$$

End Quiz

Solution to Quiz: We have

$$\begin{aligned}
2x^2 - 3x + 5 &= 2\left[x^2 - \frac{3}{2}x\right] + 5 \\
&= 2\left[x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right] + 5 \\
&= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}\right] + 5 \\
&= \left[2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8}\right] + \frac{40}{8} \\
&= 2\left(x - \frac{3}{4}\right)^2 + \frac{31}{8}
\end{aligned}$$

End Quiz